General Relativity and Quantum Mechanics: Towards a Generalization of the Lambert W Function

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Abstract

Herein, we present a canonical form for a natural and necessary generalization of the Lambert W function, natural in that it requires minimal mathematical definitions for this generalization, and necessary in that it provides a means of expressing solutions to a number of physical problems of fundamental nature. In particular, this generalization expresses the exact solutions for general-relativistic self-gravitating 2-body and 3-body systems in one spatial and one time dimension. It also expresses the solution to a previously unknown mathematical link between the linear gravity problem and the quantum mechanical Schrödinger wave equation.

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1 Introduction

The Lambert W function satisfying $W(t)e^{W(t)} = t$ was first introduced by Johann Heinrich Lambert (1728-1777), a contemporary of Euler. Though it is more than two hundred years old, its importance and universal application was only realized in the last decade of the 20th century. With the combined efforts of Gonnet and others, the W function has become a tool. Ironically, it had been injected into the Maple Computer Algebra system by Gaston Gonnet over many objections because it did not appear as a "standard" special function known in literature (e.g. see [1–3]) though it was useful as a means of expressing solutions to transcendental algebraic equations. A presentation of the work of Scott *et al.* [4] in 1992 showed that it expressed an exact solution to a fundamental problem in quantum mechanics.

This encouraged Corless *et al.* to make a literature search of the W function to find that it had been "invented" and "re-invented" at various moments in history. Its applications were numerous [5]. For

example, the W function has appeared in electrostatics, statistical mechanics (e.g. [6]), general relativity, inflationary cosmology (e.g. [7]), radiative transfer, Wien's Displacement Law of blackbody radiation (e.g. [8]), quantum chromodynamics, combinatorial number theory, fuel consumption and population growth (see e.g. [9] and references herein) etc. Within the past decade, the Lambert W function has been embedded in other computer algebra systems, multiplying the applications of this function and also increasing its awareness and thus vindicating its appearance within a Mathematical software system.

More recently, the Lambert W function has also appeared in the "lineal" gravity two-body problem [10] as a solution to the Einstein Field equations in $(1 + 1)$ dimensions. The Lambert W function appears as a solution for the case when the two-bodies have exactly the same mass. However, the case of unequal masses required a *generalization* of Lambert's function [10, eq.(81)]. Subsequent discussion brought the realization that this generalization for unequal masses had a one-to-one relationship with the problem of unequal charges for the quantum mechanical problem of Scott *et al.*

This realization fueled the impetus for investigation into a proper generalization, the focus of this article. Moreover, it became clear that something vital about the W function had been *missed*. The information and awareness of the literature on Lambert's function is still fragmentary. For example, long before D.E.G. Hare [5, (a)] extended the definition of the W function into the complex plane, such an analysis had already been done indirectly by Byers Brown [11, 12] not only for the standard W function but also for its generalization discussed herein. We see that no matter how exhaustive a literature search is made, it cannot address or cover all the aspects of a function over two hundred years old!

The goal of the present work is to examine this generalization while clarifying a number of issues with regards to the unfortunate "fragmentation" of information and awareness of Lambert's function. Of course, one can define generalizations in myriad ways. Thus, we seek a generalization that is "natural" i.e.

1. It is economical in that minimizes the need for new mathematical definitions.
2. It has applications in nature. Better still, it is ubiquitous to nature, not unlike the standard Lambert W function itself.
3. It expresses solutions to a broad range of mathematical problems.
4. Its capacity for further generalization and its reduction and correspondence to the standard Lambert W function is transparent.

Such a function satisfying these criteria is clearly a fundamental mathematical structure worthy of consideration in the Mathematical/Physical literature.

In this work, we present a canonical form for a generalization of the W function which satisfies this criteria. This is done as follows. First, we then re-examine the quantum mechanical problem of Scott *et al.* whose solution also expresses the solution to the linear gravity problem of Mann and Ohta [10]. We solve the case of unequal charges (unequal masses for the gravitational problem) intuitively. The impetus is partially derived from the notion of P.A.M. Dirac that a sound mathematical structure has a potential basis in reality (and the converse might just be true!).

Next we find that the generalization fits into a *tetration* framework (or iterative exponentiation), and requires only a *nesting* of the definitions of the Lambert W function, thus satisfying the first requirement. Next, we seek solutions to the gravitational three-body problem in $(1 + 1)$ dimensions and find that the first generalization can be naturally extended further. Finally, it is found that the end result expresses solutions to a huge class of *delayed* differential equations. It is also helpful in expressing the solution to the (three-dimensional) hydrogen molecular ion i.e. the quantum-mechanical three-body problem for the case of clamped nuclei. Concluding comments are made at the end.

2 One-Dimensional Quantum Problem

The one-dimensional version of the hydrogen molecular ion H_2^+ [11, 12, 14] is given by the double Dirac delta function model:

$$-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - q[\delta(x) + \lambda\delta(x-R)]\psi = E(\lambda)\psi \tag{1}$$

where $Z_A = q$ and $Z_B = \lambda\, q$. The ansatz for the solution has been known since the work of Frost [15]:

$$\psi \;=\; Ae^{-d|x|} + Be^{-d|x-R|} \tag{2}$$

where $0 < R < \infty$. All quantities are real. Matching of ψ at the peaks of the Dirac delta functions at $x = 0, R$ yields:

$$\left| \begin{array}{cc} q-d & qe^{-dR} \\ q\lambda e^{-dR} & q\lambda - d \end{array} \right| = 0 \tag{3}$$

and the energies are given by $E_\pm = -d_\pm^2/2$ where $d_\pm$ is governed by the secular determinant of eq. (3) when it is set to equal zero ([4, eq.(17)] for $q = 1$):

$$d_\pm(\lambda) \;=\; \tfrac{1}{2}q(\lambda+1) \pm \tfrac{1}{2}\left\{q^2(1+\lambda)^2 - 4\,\lambda q^2[1-e^{-2d_\pm(\lambda)R}]\right\}^{1/2} \tag{4}$$

When $\lambda = 1$, the pseudo-quadratic in (4) reduces to:

$$d_\pm = q[1 \pm e^{-d_\pm R}] \tag{5}$$

Although, the above has been known for more about half a century, it was not until the work Scott *et al.* [4] that the solution for $d_\pm$ was exactly found to be:

$$d_\pm = q \;+\; W(0, \pm qRe^{-qR})/R \tag{6}$$

where $\pm$ represent respectively the symmetric or *gerade* solution and the anti-symmetric or *ungerade* solution. The first argument of the W function, being zero, reminds us that Lambert's function has an infinite number of branches and that we are selecting the principal branch.

The anti-symmetric case, d_- is interesting because it appears to go to zero as $R \;\rightarrow\; 1$ for $q \;=\; 1$ [1], in other words the energy goes to zero and the corresponding eigenstate appears to go into the continuum. However, for $R < 0$, $W(-1, -Re^{-R})$ which has an order 2 branch at $R = 1$ yields a real number. Given the analysis in the complex plane of the energy eigenstates [11, 12], we can see there was already awareness of more than one branch for the solution as far back as the 1970s by mathematical physicists well versed in the mathematical framework of linear molecules.

So far, the Lambert W function could only express the solution for the case of equal charges. We now examine the general case of unequal charges. The pseudo-quadratic of eq. (4) seems complicated until we rewrite it in a very simple form *(for $q = 1$)*:

$$e^{-2xR} \;=\; \frac{(1-x)\;(\lambda-x)}{\lambda} \quad \text{where} \quad x = d_\pm\,. \tag{7}$$

The above encapsulates both *"gerade"* and *"ungerade"* solutions[2]. It also represents a canonical form for a whole class of transcendental algebraic equations. The right side of (9) is a quadratic polynomial in x only while the left-hand side is a function of x and R. It must be emphasized that R is a constant in the range $[0, \infty)$ consequently allowing infinite choices for R. The left-hand side of (7) is a whole parameter family of curves in x while the right-hand side represents only one curve in x. Therefore, eq. (7) is a

[1]For simplicity, we set the charge $q = 1$ for the rest of this work.

[2]Symmetry is lost when $\lambda \neq 1$ and the terms "gerade" and "ungerade" no longer have the same meaning.

canonical form for an *implicit* equation for x. Note that when $\lambda = 1$, we have a double root for this polynomial and both sides of (7) *factors* into two possible cases where the solutions are given by eq. (6) for $q = 1$. The problem in linear gravity [10] namely eq. (82) of ref. [10]:

$$y^2 = a^2 + \left(x^2 - a^2\right) \exp\left(2x\right) \exp\left(-2y\right)$$

relates exactly to eq. (7) by the following transformation:

$$\lambda = \frac{2x}{x+a} - 1 \;\text{ where }\quad R = -\left(x+a\right) \quad \text{and } \; d = \frac{x-y}{x+a} \tag{8}$$

when $q = 1$. The case $\lambda \neq 1$ represents the generalization we seek. Thus, we seek a solution to:

$$e^{-2xR} \;=\; a_o\, b_o (x - r_1)(x - r_2) \tag{9}$$

where in relation to the above problems $\{r_1, r_2\} = \{1, \lambda\}$ are the real roots of a quadratic polynomial and where $\{a_o, b_o\} = \{1, 1/\lambda\}$. However, we treat these parameters generally while making all necessary assumptions to ensure a real solution. Since a quadratic is merely a product of first order polynomials, this guides us intuitively to consider the following. We assume there exists a value y such that:

$$e^{-Rxy} \;=\; a_o\left(x - r_1\right) \tag{10}$$

$$e^{-Rx(2-y)} \;=\; b_o\left(x - r_2\right) \tag{11}$$

Multiplication of the left sides and right sides of eqs.(10) and (11) yields (9) the equation we desire to solve. However, individually eqs.(10) and (11) can be solved using the standard Lambert W function:

$$a_o(x_1 - r_1) \;=\; a_o \frac{W(yRe^{-r_1 yR}/a_o)}{yR} \tag{12}$$

$$b_o(x_2 - r_2) \;=\; b_o \frac{W((2-y)Re^{-r_2(2-y)R}/b_o)}{(2-y)R} \tag{13}$$

Letting $y = 1 + \epsilon$, we seek y such that $x_1 = x_2$. Thus, the "separation" parameter is governed by:

$$(r_1 - r_2) = \frac{W((1-\epsilon)Re^{-r_2(1-\epsilon)R}/b_o)}{(1-\epsilon)R} - \frac{W((1+\epsilon)\,Re^{-r_1(1+\epsilon)R}/a_o)}{(1+\epsilon)\,R} \tag{14}$$

Substituting eqs.(12) and (13) into (9)

$$e^{-2xR} \;=\; a_o b_o \frac{W\left((1+\epsilon)Re^{-r_1(1+\epsilon)R}/a_o\right)\, W\left((1-\epsilon)Re^{-r_2(1-\epsilon)R}/b_o\right)}{(1+\epsilon)(1-\epsilon)R^2} \tag{15}$$

and taking logarithms on both sides of (15) allows us to isolate an expression for x subject to the constraint that y is such that $x_1 = x_2$. Looking at "canonical" forms of eqs.(12) and (13) in comparison with (15) makes us infer the *generalized* Lambert W function as

$$\Omega_2 \;=\; \Omega_2(a_o, b_o, r_1, r_2, R) \;=\; W(z_1)\, W(z_2) \tag{16}$$

where

$$z_1 \;=\; (1+\epsilon)Re^{-r_1(1+\epsilon)R}/a_o$$

$$z_2 \;=\; (1-\epsilon)Re^{-r_2(1-\epsilon)R}/b_o$$

and where $\epsilon = \epsilon(a_o, b_o, r_1, r_2, R)$. The above is a product of standard Lambert W functions in the same fashion a quadratic polynomial is the product of first order polynomials. When $r_1 = r_2$, it is clear that $\epsilon = 0$ (or when an analytical expression for ϵ is possible) and we recover the solution in terms of the standard Lambert W function. The subscript 2 on Ω reminds us that the right side is a second order

polynomial. In general, y or for that matter ϵ will be referred to as a *separation* parameter, which allows the generalized function shown here to be decoupled as a product of standard Lambert W functions. Note that when $a_o = b_o = 1$, we recover the symmetric (gerade) solution and with $a_o = b_o = -1$, the anti-symmetric (ungerade) solution of eq. (6).

In view of eq. (14), one realizes that the *separation parameter* ϵ is itself governed by a transcendental equation which looks even more complicated than the original transcendental algebraic equation of eq. (7). The critic then asks: how are we further ahead? Part of the answer lies in considering an important aspect about generalizations i.e. whether or not they have a capacity to collapse into special cases other than the original function from which the generalization was inferred or collapse into previously unknown solutions. Let a_o, b_o, r_1, r_2 be the values as quoted below (9) and let us make a series expansion of $x_1 - x_2$ in the parameter λ:

$$\begin{aligned} x_1 - x_2 \quad &\approx \quad \frac{1 + e^{W\left(R(1+\epsilon)e^{-R(1+\epsilon)}\right)} e^{R(1+\epsilon)}}{e^{W\left(R(1+\epsilon)e^{-R(1+\epsilon)}\right)} e^{R(1+\epsilon)}} - 2\,\lambda - 2\,R\,(-1+\epsilon)\ \lambda^2 \\ &-4\,R^2\,(-1+\epsilon)^2\,\lambda^3 - \frac{28}{3}\,R^3\,(-1+\epsilon)^3\,\lambda^4 - 24\,R^4\,(-1+\epsilon)^4\,\lambda^5 + O\left(\lambda^6\right) \qquad (17) \end{aligned}$$

By inspection, we can see that $\epsilon = 1$ eliminates all terms of order λ greater than 1 and consequently:

$$x_1 - x_2 \ = \ \frac{1 + e^{W(2Re^{-2R})} e^{2R}}{e^{W(2Re^{-2R})} e^{2R}} - 2\,\lambda$$

Solving for λ such that the difference $x_1 - x_2$ is zero, yields:

$$\lambda \ = \ \frac{1}{2} + \frac{W(2Re^{-2R})}{4R} \qquad (18)$$

In this case, the solution for x is found to be:

$$x \ = \ -\frac{1}{2R} \ln\left(\frac{W(2Re^{-2R})}{2\,R}\right) \qquad (19)$$

Thus for λ satisfying (18) and $\epsilon \to 1$, we have a previously unknown solution to eq. (7) in terms of the standard Lambert W function. Granted $\epsilon \to 1$ represents a limiting extreme: this solution has been vindicated by numerical and analytical demonstrations using computer algebra. Thus, we have a previously unknown particular solution for the case of unequal charges (or unequal masses in the linear gravity problem) but for a peculiar value of λ dependent upon R which admittedly is not physically useful since the physical charges Z_A and Z_B are constants independent of R. This bears a striking resemblance of the results of Demkov who found analytical solutions for the three-dimensional hydrogen molecular ion H_2^+ but for a particular choice of charges (again not physically useful for the same reasons) which reduced to Whittaker functions i.e. the type of solutions found for the hydrogen atom [16].

At any rate, this demonstration shows that the generalization we inferred is not impotent but can also collapse into simpler special functions for special cases. There are other such cases. Since eq.(14) has the form $r_1 - r_2 = f(r_2) - g(r_1)$, one particular obvious solution is simply $r_1 = f(r_2)$ and $r_2 = g(r_1)$. Decoupling this particular solution with respect to r_1 and r_2 yields equations governing the latter:

$$\begin{aligned} r_1 \quad &= \quad \frac{W(R(1-\epsilon)\exp(\frac{(\epsilon-1)W\left((1+\epsilon)e^{-r_1 R(1+\epsilon)}/a_o\right)}{1+\epsilon})/b_o)}{R(1-\epsilon)} \qquad (20) \\ r_2 \quad &= \quad \frac{W(R(1+\epsilon)\exp(\frac{(1-\epsilon)W\left((1-\epsilon)e^{-r_2 R(1-\epsilon)}/b_o\right)}{1-\epsilon})/a_o)}{R(1+\epsilon)} \end{aligned}$$

At this stage, we can change our context: rather than look for solutions of eq. (9) for arbitrary choices of the parameters R, a_o, b_o, r_1 and r_2 and be burdened with the analytical determination of ϵ: we instead use

ϵ as a *common* parameter to r_1 and r_2 allowing us to find which values of r_1 and r_2 satisfy eq. (9) for a given choice of the remaining parameters a_o b_o and R. Moreover, we also find the following previously unknown *exact* solutions to (9):

$$r_1 = \frac{1}{b_o}, \quad a_o = \frac{e^{\frac{-2R(1+b_o r_2)}{b_o}}}{r_2}, \quad x = \frac{1+b_o r_2}{b_o}, \quad \epsilon \to 1\,. \tag{21}$$

where r_2 and b_o are arbitrary real numbers and also

$$r_2 = \frac{1}{a_o}, \quad a_o = -\frac{2R}{2r_1 R + \ln(r_1 b_o)}, \quad x = -\frac{1}{2\,R}\,\ln(r_1 b_o)\,, \tag{22}$$

where r_1 and b_o are arbitrary real numbers. Eq. (21) is in *closed* form (elementary functions) and (22) is in terms of the Lambert W function which bears some resemblance to eq. (19) and could be called another “Demkov”-type solution. This is one of the utilities of the “separation” parameter ϵ: eqs. (14) and especially (20) look more complicated than the original equation (9), they are however useful in finding its particular solutions[3]. The parameter ϵ also illustrates the reduction from the proposed generalized function to the simpler (standard) Lambert W function in a transparent manner. In the next section, we properly define the function Ω_2 which we derived intuitively.

3 Iterated Exponentiation

Infinitely iterated exponentiation or *tetration* is defined as the limit,

$$\lim_{n\to\infty} {}^{\alpha}n \;=\; \alpha^{\alpha^{\alpha^{\cdot^{\cdot^{\cdot}}}}} \;\equiv\; \mathfrak{H}(\alpha) \tag{23}$$

This function can be written compactly as:

$$\mathfrak{H}(\alpha) = e^{-W(-\ln\alpha)} \qquad \alpha \in \mathbb{C} \tag{24}$$

The product on n W functions is given by:

$$W(z_1)W(z_2)\ldots W(z_n) = (z_1 z_2 \ldots z_n)e^{-[W(z_1)+W(z_2)+\ldots+W(z_n)]}$$

following from the defining relation $z = W(z)e^{W(z)}$. The Lambert W function is governed by the addition law [17]:

$$W(a) + W(b) \;=\; W(ab\;[1/W(a) + 1/W(b)]) \tag{25}$$

for $\Re(a)$ or $\Re(b) > 0$. By induction, the product is then:

$$\begin{aligned} W(z_1)\,W(z_2)\ldots W(z_n) \;&=\; (z_1 z_2 \ldots z_n)\,e^{-W(f_n)} \\ &=\; (z_1 z_2 \ldots z_n)\,W(f_n)/f_n \end{aligned} \tag{26}$$

where f_n stands for an expression involving W functions of z_1, $z_2 \ldots z_n$ that follows from the addition law. E.g. for $n = 2$ and $n = 3$,

$$\begin{aligned} f_2 \;&=\; z_1 z_2\,(1/W(z_1) + 1/W(z_2)) \\ f_3 \;&=\; f_2 z_3\,(1/W(f_2) + 1/W(z_3)) \end{aligned}$$

[3] We anticipate the existence of other cases when ϵ can be solved in closed form but of course, in general, this is not the case. Indeed, ϵ might not even exist which is why the generalization is needed.

Thus,

$$
\begin{aligned}
W(z_1)\,W(z_2) &= z_1 z_2\, W(f_2)/(z_1 z_2 (1/W(z_1)\; +\; 1/W(z_2))) \\
&= W(f_2)/(1/W(z_1)\; +\; 1/W(z_2)).
\end{aligned} \tag{27}
$$

In the context of the tetration function $\mathfrak{H}(\alpha)$ of (24), the *product of Lambert W functions* is really just:

$$
\Omega_n(z_1, z_2, \ldots, z_n) = W(z_1)W(z_2)\ldots W(z_n) = (z_1 z_2 \ldots z_n)\mathfrak{H}(e^{-f_n}) \tag{28}
$$

a multiple of the continued tetration function. To reiterate, for eqs.(12) and (13) when $x = x_1 = x_2$, the solution to eq. (9) is given by a product of W functions, as one substitutes x_j into $(x - r_j)$ for $j = 1, 2$. Hence, the solution is in terms of one W function, albeit nested. In this framework, thanks to the addition law for the W function, no generalization in the sense of additional mathematical definitions is needed: it is simply one single W function evaluated at a point involving other W functions.

Furthermore, if we use eq. (28) for $n = 2$ and combine with the earlier result in (16), we obtain a simple yet general relation between the separation parameter ϵ and x:

$$
(r_2 - r_1)\epsilon \;=\; \frac{W(f_2)}{R} \;+\; (r_1 + r_2) - 2\,x \tag{29}
$$

which is consistent with eq. (15) in view of the addition theorem for W functions. We can already identify regimes. For example, for real positive roots r_1 and r_2 the term $W(f_2)$ is bounded in many cases and consequently[4] $\lim_{R\to\infty} \epsilon = \pm 1$.

Although we initially assumed $\Re\, z_n$ is not a negative number, injecting of numbers into eq. (28) show that this identity formula is very robust after all if one considers different W functions on different branches. For example, (28) holds if $x_1 = 1 + 2i$, $x_2 = -1$ for $W(x_2) = W(-1, x_2)$ and is complex-valued.

4 Rational Polynomials

In the previous sections, we considered the right side of the transcendental equations to be polynomials, but one can also consider a rational polynomial. Let us consider the equation where the right side is a ratio of first order polynomials:

$$
e^{-2Rx} \;=\; \frac{a_o(x - r_1)}{b_o(x - r_2)} \tag{30}
$$

In parallel to what we did before, we can consider:

$$
\begin{aligned}
e^{-Rxy} &= a_o\,(x - r_1) &(31)\\
e^{-Rx(2-y)} &= \frac{1}{b_o\,(x - r_2)} &(32)
\end{aligned}
$$

Eq.(32) can also be solved in terms of the standard Lambert W function i.e. once the inverse is taken on both sides of (32) i.e. $(2 - y) \to (y - 2)$ in eq. (11). The generalization is thus:

$$
\Omega_{1,-1} \;=\; W\left((1+\epsilon)Re^{-r_1(1+\epsilon)R}/a_o\right)\; W\left(-(1-\epsilon)Re^{+r_2(1-\epsilon)R}/b_o\right) \tag{33}
$$

where the subscripts $1, -1$ denote respectively the polynomial degrees in (30). This also fits into the tetration framework of section 3, the only change being done to the argument x_2 and the right side of (30) can be generalized to a rational polynomial of higher order as both numerator and denominator can be "separated" in a manner demonstrated in eqs.(31) and (32) to yield generalized Ω functions that can be further nested together using the addition theorem. The next section presents applications of this type of problem.

[4]Of course, we realize that it is much easier to solve the transcendental equations numerically; the goal of this exercise is to define and justify our generalization of the W function.

5 Three-Body Linear Gravitational Motion

The solution of the three-body one $(1+1)$ dimensions via dilation theory requires solving for V which is governed by the following equation [18, eq.(32)]:

$$(V-m_1)\,(V-m_2)\,(V-m_3) \tag{34}$$

$$\begin{aligned} &= m_1\, m_2\,(V - s_1\, s_2\, m_3)\ \exp(K\,V\,R\,|\sin(q)|) \\ &+ m_1\, m_3(V + sq\, s_2\, m_2)\ \exp(K\,V\,R\,|\sin\left(q+\frac{\pi}{3}\right)|) \\ &+ m_2\, m_3\,(V - sq\, s_1\, m_1)\ \exp(K\,V\,R\,|\sin\left(q-\frac{\pi}{3}\right)|) \end{aligned} \tag{35}$$

where

$$\begin{aligned} s_1 &= \quad \text{sign}(\sin(q+\tfrac{\pi}{3})) \\ s_2 &= - \ \text{sign}\,(\sin(q-\tfrac{\pi}{3})) \\ sq &= \quad \text{sign}\,(\sin(q)) \end{aligned}$$

This problem has no closed form solution. The hard part is in trying to extricate the exponential terms from the rest of the expression. At $q = 0$ or $\pm\pi/3$, the trigonometric quantities s_i for $i = 1, 2$ and sq simplify to specific values. E.g. for $q = 0$ and $V \neq 0$, it is found that:

$$\exp(-2R_t\,V) \;=\; \frac{1}{m_3(m_1+m_2)}\,(V-m_3)(V-(m_1+m_2)) \tag{36}$$

where $R_t = KR\sqrt{3}/4$. As we can see V is governed by exactly the same type of transcendental equation as eq. (9) whose solution, as we have seen, can be expressed in terms of our Ω_2 function. In the case when $m_3 = m_1 + m_2$, we have a double root and thus a solution for (36) can be expressed exactly in terms of the standard W function and has two solutions:

$$V \;=\; m_3 - \frac{1}{R_t}W\left(\pm\, R_t\, m_3 e^{R_t m_3}\right) \quad \text{when} \quad q=0,\ m_3 = m_1+m_2 \tag{37}$$

Very similar solutions also exists for $q = \pi/3$ with $m_1 = m_2 + m_3$ and for $q = -\pi/3$ and $q = 2\pi/3$ with $m_2 = m_1 + m_3$. This demonstrates that at every $\pi/3$, solutions in terms of our generalized Ω functions exist. At $q = \pi/6$ and equal masses i.e. $m_1 = m_2 = m_3$ we arrive at two possible equations depending on the outcome of factorization. One equation is:

$$\exp(cV) = -\frac{(V-m)}{m}\,, \tag{38}$$

and the other equation is:

$$\exp(cV) = -\frac{(V-m)^2}{m(V+m)}\,, \tag{39}$$

where $c = KR/2$. Eq.(38) can be solved in terms of the standard W function i.e.

$$V \;=\; m\left(1 - \frac{W(mce^{mc})}{mc}\right)$$

which is in the same form as the range x_{range} and drift z_{drift} equations of section 2 and the anti-symmetric solution d_- in section 3 with the exact correspondence given by $\eta = -mc$ and $b = v_{\text{x},0}$. However the right side of (39) involves a rational polynomial of the form $P_2(V)/Q_1(V)$ and requires $\Omega_{2,-1}$ to express the solution. If we consider the region $0 < q < \pi/3$, we obtain the form:

$$\exp(-c\,\sqrt(3)\ \cos(q)V) = \frac{P_N(V)}{Q_M(V)} \quad \text{where} \quad M, N \to \infty \tag{40}$$

This rational polynomial can be generated from:

$$\frac{f\,(V-m_3)\,(V2 - V*(m_1+m_2) - m_1*m_2(f^2-1))}{m_3*(m_1*m_2*(f^2-1)+m_1*f^2+m_2*V)} \rightarrow \frac{P_N(V)}{Q_M(V)}$$

where $f = \exp(c\,\sin(q)V)$ generates the rational polynomials. In this regime $\sin(q) \approx q$ is small and $\cos(q) \approx 1$ and thus this treatment is possible. A similar story applies to other regimes $[\pi/3, 2\pi/3]$, $[2\pi/3, \pi]$ etc ... Thus, we can see that the general solution falls in the following general form.

6 General Form

Thus, the fully generalized form concerns expressing solutions to this general class of transcendental algebraic equation:

$$e^{\pm k\,x} \;=\; \frac{P_N(x)}{Q_M(x)} \tag{41}$$

where $k > 0$ is a constant and $P_N(x)$ and $Q_M(x)$ are polynomials in x of respectively orders N and M. The general solution can be expressed by $\Omega_{N,M}$ which is formally a product of $N+M$ (standard) Lambert W functions with $N+M-1$ "separation" constants.

The standard W function applies for cases when $N=1$ and $M=0$ and expresses solutions for the case of equal charges for eq. (1) or the case of equal masses for the lineal two-body $(1+1)$ gravity problem. Correspondingly, the case $N=2$ and $M=0$ expresses solutions for the case of unequal charges (or unequal masses for the lineal two-body gravity problem) and some particular cases of the lineal three-body gravity problem. In the limit as $M, N \rightarrow \infty$, this equation can be used to express solutions of the three-body lineal gravity problem as we have just shown.

Moreover, the case $N=2$ and $M=0$ (and more generally $N=2$ and $M=1$) express the solutions of a significant class of delayed differential equations [19, eq.(3)]. These arise in a variety of mechanical or neuro-mechanical (oscillatory) systems in which non-linear feedback plays an important role. These have applications in e.g. models for physiological systems (medicine) [20].

Recently Adilet Imambekov and Eugene Demler [21] considered a bose-fermi mixture in one dimension. We note that their Eq.(2):

$$H = -\sum_i^N \frac{\partial^2}{\partial x_i^2} + 2c\sum_{i<j}\delta(x_i - x_j) \quad c > 0$$

is in fact the Schrödinger wave equation for a *linear molecule*. Their analytical solution as written in their eq. (20) is indeed a special case of the general form expressed (41). Thus, we can also see that this function has fundamental applications in Physics and can play a fundamental role in Mathematics.

7 Conclusions

Thus we have identified a generalization of the Lambert W function or Omega function as solutions to a large class of *transcendental equations* as written in eq. (41). This generalization, denoted Ω_n, satisfies the criteria mentioned earlier in our introduction by using the *analytic* framework of tetration.

We have also shown that the two-body problem in lineal gravity and double well linear quantum mechanics have the same generalization of W, namely Ω_2 as solutions. The reason why is because the linear gravity

theory via dilaton theory produces a partial differential equation, namely eq. (30) of ref. [10] which can be treated formally as the Schrödinger wave equation as written in (1). This is to be elaborated elsewhere [22].

Furthermore, this work on the Lambert W function has helped in finding analytic solutions to the quantum mechanical 3-body problem known as the hydrogen molecular ion in the case of clamped nuclei of equal charges [23]. All this vindicates our proposed generalization of the W function as being of fundamental and physical importance.

Most of the special functions in the known literature (e.g. [1]) are special cases of the hypergeometric functions and/or the Meijer G-function [3]. The Lambert W function apparently bears no relationship to these functions and belongs to a class of its own. The generalization we have presented is a first step in identifying that class.

Although we have inferred a canonical form for a generalization as expressed by (41) and given mathematical and physical justifications for it, we have yet to clearly identify a domain and range of applicability or conditions of analytic continuation. Neither have we formulated Taylor series nor asymptotic series useful for computation. Naturally, it is expected that our “separation” parameter ϵ in (15) likely has a restricted domain of applicability (its primary use being to infer the general result starting from the standard W function).

Nonetheless, given that we have fast computational means to transcendental algebraic equations, the task of obtaining the floating-point attributes in many cases is trivial today with modern algorithms. Ironically, it has been our capacity to readily solve these transcendental equations numerically that has made many take an analytical solution for granted. This may have been a mistake as we can see that such analytical solutions are ubiquitous to certain fundamental problems in Physics and Mathematics. In a true sense, this work is only a beginning.

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (9th printing), Dover, New York, 1970.

[2] G. B. Arfken, *Mathematical Methods for Physicists*, 2nd ed., Academic Press, New York, 1970.

[3] Y. L. Luke, *The special functions and their approximations*, Vol. 1, Academic Press, New York, 1969.

[4] T. C. Scott, J. F. Babb, A. Dalgarno, and J. D. Morgan III, *The Calculation of Exchange Forces: General Results and Specific Models*, J. Chem. Phys. **99**, (1993), 2841-2854.

[5] (a) R. Corless, G. Gonnet, D. E. G. Hare and D. Jeffrey, *Lambert's W Function in Maple*, MapleTech **9**, ed. T. C. Scott, (Spring 1993); (b) R. Corless, G. Gonnet, D. E. G. Hare, D. Jeffrey and D. Knuth, *On the Lambert W Function*, Advances in Computational Mathematics, **5**, (1996), 329-359.

[6] J.M. Caillol, *Some applications of the Lambert W function to classical statistical mechanics*, J. Phys. A.: Math. Gen. ttextbf36, (2003), 10431-10442.

[7] A. Ashoorioon, A. Kempf and R.B. Mann, Phys.Rev. **D71**, (2005), 023503; R. Easther, B.R. Greene, W.H. Kinney and G. Shiu Phys. Rev. **D67** (2003), 063508.

[8] S. R. Valluri, D. J. Jeffrey and R. M. Corless, Some applications of the Lambert W function to Physics, Can. J. Phys.**78**, (2000), 823-831;
see also Eric Weisstein's World of Physics, (22.02.2005)
`http://scienceworld.wolfram.com/physics/WiensDisplacementLaw.html`

[9] S. R. Cranmer, *New views of the solar wind with the Lambert W function,* Am. J. Phys. **72**, (2004), 1397-1403.
`http://cfa-www.harvard.edu/˜scranmer/News2004/`

[10] R. B. Mann and T. Ohta, *Exact solution for the metric and the motion of two bodies in* $(1+1)$-*dimensional gravity*, Phys. Rev. D. **55**, (1997), 4723-4747.

[11] P. R. Certain and W. Byers Brown, *Branch Point Singularities in the Energy of the Delta-Function Model of One-Electron Diatoms*, Intern. J. Quantum Chem. **6**, (1972), 131-142.

[12] W. N. Whitton and W. Byers Brown, *The Relationship Between the Rayleigh-Schrödinger and Asymptotic Perturbation Theories of Intermolecular Forces*, Int. J. Quantum Chem. **10**, (1976), 71-86.

[13] E. Packel and D. Yuen, *Projectile Motion with Resistance and the Lambert W Function*, The College Math. Journal **35**, no. 5, (Nov. 2004), 337-350.
`http://math.lfc.edu/˜packel/PackelYuenProjectilePaper.pdf`

[14] D. R. Herrick and F. H. Stillinger, Phys. Rev. A **11**, (1975) 42.

[15] A. A. Frost, *J. Chem. Phys* **25**, (1956), 1150.

[16] Yu. N. Demkov and I. V. Komarov, *Hypergeometric partial solutions in the problem of two Coulomb centers*, Author affiliation: Leningrad State Univ., Leningrad, USSR Teoreticheskaya i Matematicheskaya Fizika (USSR) (trans. Theoretical and Mathematical Physics) **38**, no.2, (1979), 174-176.

[17] `http://functions.wolfram.com/ElementaryFunctions/ProductLog/16/02/01/`

[18] J. J. Malecki and R. B. Mann, *3-Body Dynamics in a* $(1+1)$ *Dimensional Relativistic Self-Gravitating System*, Phys. Rev. E. **69**, (2004), 1-26.
`http://arxiv.org/pdf/gr-qc/0306046`

[19] S. A. Campbell, *Stability and bifurcation in the Harmonic Oscillator with multiple, delayed feedback loops*, Dynamics of Continuous Discrete and Impulsive Systems, **5**, (1999), 225-235.

[20] N. MacDonald, *Biological Delay Systems: Linear Stability Theory*, Cambridge University Press, 1969; U. an der Heiden, *Delays in physiological system*, J. Math. Biol. **8**, (1979), 345-364.

[21] A. Imambekov and E. Demler, *Exactly solvable one-dimensional Bose-Fermi mixture*, private communication, 2005.

[22] P. S. Farrugia, R. B. Mann, and T. C. Scott, *N-body Gravity and the Schroedinger Equation* in preparation, 2006.

[23] T. C. Scott, M. Aubert-Frécon and J. Grotendorst, *New approach for the electronic energies of the hydrogen molecular ion*, Chem. Phys. **324**, (2006), 323-338.